Comparison of Numerical Schemes for Modelling Supercritical Flows along Urban Floodplains

by

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Introduction

• Urban flood modelling:
  – 1D
  – 2D
  – 1D/2D
  – 3D
Governing Equations

• The Saint Venant Equations

\[ \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \]

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \left( \frac{\partial h}{\partial x} - S_0 + S_f \right) = 0 \]

In characteristic form

\[ \frac{\partial}{\partial t} (u + 2c) + c(Fr + 1) \frac{\partial}{\partial t} (u + 2c) = 0 \]

\[ \frac{\partial}{\partial t} (u - 2c) + c(Fr - 1) \frac{\partial}{\partial t} (u - 2c) = 0 \]
Governing Equations

• In vector form

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{R}
\]

where

\[
\mathbf{U} = (A, Q)^T
\]

\[
\mathbf{F} = (Q, \frac{Q^2}{A} + g \frac{A^2}{2b})^T
\]

\[
\mathbf{R} = (0, gA(S_0 - S_f))^T
\]

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{J} \frac{\partial \mathbf{U}}{\partial x} = 0
\]

\[
\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix}0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}
\]
Characteristic slopes

- The characteristics of the above equations are given by

\[ C^\pm = \frac{Q}{A} \left(1 \pm \frac{1}{Fr}\right) \]

\[ \frac{dx}{dt} = u \pm \sqrt{gh} \]

Subcritical Flow \( u < \sqrt{gh} \) \( Fr < 1 \)

Supercritical Flow \( u > \sqrt{gh} \) \( Fr > 1 \)
Boundary condition issues

• Boundary problem – mixed flows
  – Subcritical to supercritical - a total of three boundary conditions
  – Supercritical to subcritical – only one boundary condition

• Tackling the issue
  – Full equation (explicit scheme)
  – Partially reduced CAT
  – Completely reduced CAT
MIKE11

- Modified momentum equation (MIKE11 Reference Manual)

\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q
\]

\[
\frac{\partial Q}{\partial t} + \beta \frac{\partial (\alpha \frac{Q^2}{A})}{\partial x} + gA \frac{\partial h}{\partial x} + \frac{gQ|Q|}{C^2AR} = 0
\]

Default suppression

\[
\beta = \begin{cases} 
1 - Fr^2 & \text{for } Fr \leq 1 \\
0 & \text{for } Fr > 1 
\end{cases}
\]

Slopes of the characteristic curves (for \( Fr > 1 \)) are:

\[
\frac{dx}{dt} = \pm \sqrt{gh}
\]
MIKE11

• The solution in MIKE11 is based on a 6-point (Abbott-Ionescu) implicit finite difference scheme
  – in which a staggered computational grid of alternating $Q$ (discharge) and $h$ (water level) points is used and solved.
Kutija Approach

• The de Saint Venant equations can be written in the form (Kutija 1993)

\[
\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0
\]

\[
\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2h)}{\partial x} + gh \frac{\partial h}{\partial x} = 0
\]

• Split up the convective momentum term and reduce only a part of it

\[
\frac{\partial (uh)}{\partial t} + u \frac{\partial (uh)}{\partial x} + gh \frac{\partial h}{\partial x} = 0
\]

• Then, slopes of the characteristic curves are

\[
\frac{dx}{dt} = \frac{u}{2} \pm \sqrt{\frac{u^2}{4} + gh}
\]
Kutija Approach

• Assuming a rectangular cross-section, the de Saint Venant equations in this approach are given by

\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0
\]

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) - Q \left[ \frac{\partial}{\partial x} \left( \frac{Q}{A} \right) + \frac{Q}{Ab} \frac{db}{dx} \right] + gA \left( \frac{\partial h}{\partial x} - S_0 + S_f \right) = 0
\]

• The numerical software developed is based on a 4-point implicit finite difference scheme (Preissmann Scheme).
Roe Scheme

• First order upwind scheme (Crossley 1999; Garcia-Navarro and Burguete 2006; Roe 1981).
• Explicit scheme which solves the full equations.

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0
\]

\[
U_{i}^{n+1} = U_{i}^{n} - \frac{\Delta t}{\Delta x} (F_{i+1/2} - F_{i-1/2})
\]

\[
F_{i+1/2} = \frac{1}{2} (F_{i+1} + F_{i}) - \frac{1}{2} [(P|\Lambda|P^{-1})]_{i+1/2} (U_{i+1} - U_{i})
\]

\[
\tilde{\Lambda} = \begin{bmatrix}
\tilde{u} + \tilde{c} & 0 \\
0 & \tilde{u} - \tilde{c}
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
1 & 1 \\
\tilde{u} + \tilde{c} & \tilde{u} - \tilde{c}
\end{bmatrix}
\]
- Manning's roughness coefficient $= 0.035$
- 101 channel cross-sections defined every 50 m
- time step = depending on the courant condition for the explicit scheme and an adaptive time step for the implicit ones
Prismatic channel

- water depth based on uniform flow conditions which is defined by the bottom slope and Manning's $n$
Simulation results at $x = 2500$ m

$$S_1 = 0.01$$
$$Fr = 0.71 \text{ to } 0.79$$

$$S_2 = 0.02$$
$$Fr = 0.97 \text{ to } 1.08$$
Simulation results at $x = 2500$ m

$S_3 = 0.05$
$Fr = 1.47$ to $1.63$

- No phase error in all cases

Reducing the CATs partially or completely gives very close results compared to the results of the full equations
Non-prismatic Channel (contracted cross-section)

- Manning's roughness coefficient = 0.035
- 101 channel cross-sections defined every 50 m
- time step = depending on the courant condition for the explicit scheme and an adaptive time step for the implicit ones
Simulation results at $x = 2500$ m

\[ S_1 = 0.01 \]
\[ Fr = 0.74 \text{ to } 0.81 \]

\[ S_2 = 0.02 \]
\[ Fr = 1.01 \text{ to } 1.12 \]
Simulation results at $x = 2500 \text{ m}$

$S_3 = 0.05$
$Fr = 1.54$ to $1.70$

- No phase error in these cases also

Reducing the CATs partially or fully gives closer results compared to the results of the full equations (with only around 5mm difference in the peak flood depth by MIKE11)
Conclusions

• Considering the 1D test cases used in this study, partially or fully reducing the convective acceleration terms does not cost significantly.

• This approximation may not hold in cases such as flows characterised by steep fronted wave or in a dam-break analysis.
Thank You!